SEEING IN SPACE IS DIFFICULT: AN APPROACH TO 3D GEOMETRY THROUGH A DGE

Francesca Ferrara[1], Maria Flavia Mammana[2]

[1]Dipartimento di Matematica "Giuseppe Peano", Università di Torino, Italy
[2]Dipartimento di Matematica e Informatica, Università di Catania, Italy

In this paper, we present an approach to spatial geometry that involved a group of university students, who engaged in visual experiences while discussing about geometrical properties using a dynamic geometry environment. Drawing on aspects related to the difficulty of seeing in 3D, we introduce suitable connections between quadrilaterals and tetrahedra as a way to enhance visual skills in space geometry. In so doing, we show examples of the way learners manage "to see in space" through the affordances offered by the DGE.

INTRODUCTION

Discussing about relationships between space and geometry, Henri Poincaré (1905) pointed out that one geometry cannot be truer than another; it can only be more convenient. This much depends on our habit to work with geometrical objects in a certain way. Among all possible conventions, we are guided in choice by experience. For Poincaré, Euclidean geometry is the most convenient, because it is the simplest one, best adapting to our impressions and agreeing with properties of the natural solid bodies that we touch and see in the world around us.

3D geometry is not felt as convenient at all. At secondary school, where the study of geometry in space is expected, teachers show poor confidence about—and prefer to avoid—it, despite its relevance with respect to real world and scientific disciplines. The general reputation that spatial geometry is difficult is usually connected to the feeling that *seeing* in 3D is difficult (Bakó, 2003).

The question of *seeing* is likely to be the most important for our discussion. In fact, geometry in space involves visual challenges related to the ontological difference between three-dimensional objects and bi-dimensional diagrams that embody them. A pedagogical challenge then, is relative to studying approaches to 3D geometry that may encourage and foster seeing in space. Our study seeks to draw attention to this aspect and presents an approach to the study of spatial geometry that makes use of dynamic geometry environments (DGEs). The study is part of a wide research, whose focus is on the visual challenges involved in the study of objects in space and on the role of technology to address such challenges. To pursue our interest in seeing with respect to the use of DGEs, we anticipate that the discovery of relationships between geometric 2D and 3D figures is a crucial aspect of studying 3D geometry, and that the visual and cognitive potential of interlacing related but different figures is offered by the use of DGEs, allowing for moving back and forth between plane and space.

2014. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.) Proceedings of the Joint Meeting of PME 38 and PME-NA 36,Vol. 3, pp. 57-64. Vancouver, Canada: PME.

THEORETICAL BACKGROUND

Relevance of seeing in mathematics

Talking about his practice, Walter Whiteley highlights the central role of the visual:

> I am a research mathematician, working in discrete applied geometry. My own practice of mathematics is deeply visual: the problems I pose; the methods I use; the ways I find solutions; the way I communicate my results. The visual is central to mathematics as I experience it. It is not central to mathematics as many teachers present it nor as students witness it. This contrast is striking. (Whiteley, 1994, p. 1)

The etymology of visual comes from the Latin word *visus* that means "sight", or from *visus*, past participle of *videre*, which is "to see". That visual thinking is essential for professional mathematicians has been studied in research in mathematics education (e.g. Healy & Hoyles, 1999). For Sfard (2008), visual mediators are fundamental elements of the discursive activity in/of mathematics, and "in spite of the famous "intangibility" of mathematical objects, mathematical communication depends on what we see no less than do other, less abstract types of talk" (p. 146).

Besides the fact that mathematicians do not see the same thing in a unique diagram, what is visual for the expert mathematician/the teacher is not always like that for an apprentice/a learner. The "striking contrast" said above opens room for pedagogical intervention to make mathematics—at least partly—a visible enterprise. Presmeg (2006) has marked the emergence of "effective pedagogy that can enhance the use and power of visualization in mathematics education" (p. 227). In mathematics teaching, visual approaches frequently give a straightforward perception of the results. For example, a square number is immediately thought of—seen—as the sum of even numbers through a diagram, where a suitable disposition of elements representing a sum of even numbers forms a square.

Seeing and 3D geometry

The visual challenges involved in the study of spatial geometry are related to having to do with "flat" diagrams for geometrical figures, bi-dimensional representations of 3D objects. A study from the eighties had marked the existence of coding problems, in terms of *knowing* versus *seeing*, in the teaching of space geometry (Parzysz, 1988):

> The problems of coding a 3D geometrical figure into a single drawing have their origin in the impossibility of giving a close representation of it, and in the subsequent obligation of 'falling back' on a distant representation […] an insoluble dilemma, due to the fact that what one *knows* of a 3D object comes into conflict with what one *sees* of it. (pp. 83-84)

Moreover, perception of the third dimension greatly depends on the way we perceive depth and the cluttered space around us, and it is, in turn, a matter of how our eyes and mind measure reality and virtual reality. Different sources of information about layout can entail different fashions of measuring it and, then, of perceiving around us spaces with different geometrical natures (Cutting, 1997).

Within this perspective, we think that, in principle, the 2D diagram is far from seeing in it the 3D figure in the same measure as the 3D figure is far from diagramming it in the 2D diagram. Briefly speaking, the figural and the conceptual aspects of the figure are in conflict with each other, beyond their being in conflict with visual perception.

Seeing something *as* something else

As Douglas R. Hofstadter reports in his chapter *On Seeing A's and Seeing As*, one time, in the context of an old debate with Giancarlo Rota, Stanislaw Ulam parried:

> What makes you so sure that mathematical logic corresponds to the way we think? Logic formalizes only very few of the processes by which we actually think. The time has come to enrich formal logic by adding to it some other fundamental notions. What is it that you see when you see? You see an object *as* a key, a man in a car *as* a passenger, some sheets of paper *as* a book. It is the word 'as' that must be mathematically formalized…. (Hofstadter, 2005, p. 264)

The process of seeing something *as* something else, which was felt as impossibly hard by Rota, was on the contrary crucial for Ulam's idea of mathematical thinking as permeated with analogies between analogies—a point that is relevant for us to the extent that, drawing on Hofstadter, we think of "as" as central to "abstract seeing", in terms of seeing 3D properties in flat diagrams. We believe that such a step requires a kind of manipulation that is not easy for high school students, because their previous studies of solids were likely to involve physical manipulation. This is where we think that the use of DGEs can further occasions for new experiences of engaging with the diagrams and discovering invariants and changes. Again, we can recall Hofstadter (1997), who refers to his screen-based observations as both facts and theorems:

> To me, this result was so clearly true that I didn't have the slightest doubt about it. I didn't need a proof. If this sounds arrogant, let me explain. The beauty of *Geometer's Sketchpad* is that it allows you to discover instantly whether a conjecture is right or wrong-if it's wrong, it will be immediately obvious when you play around with a construction dynamically on the screen. If it's right, things will "stay in synch" right on the button no matter how you play with the figure. The degree of certainty and confidence that this gives is downright amazing. It's not a proof, of course, but in some sense, I would argue, this kind of direct contact with the phenomenon is even more convincing than a proof, because you really see it all happening right there before your eyes. None of this means that I did not want a proof. In the end, proofs are critical ingredients of mathematical knowledge, and I like them as much as anyone else does. I just am not one who believes that certainty can come only from proofs. (p. 10)

Starting from this background, in the next sections, we discuss an approach to spatial geometry that draws on a definitional "analogy" between two figures: quadrilateral and tetrahedron, using a DGE like Cabri Géomètre 3D. In so doing, we will show examples of what the participants manage to "see in space" through the affordances offered by the DGE.

METHODOLOGY

Participants, tasks and data collection

The participants of the study were a group of 12 university students (age 22) enrolled in a Foundations of Mathematics class in a Faculty of Mathematics in Eastern Sicily. The classroom was divided into groups of two or three students, and each group was seated in front of two computers in a computer laboratory. The computers had one Cabri II Plus and the other Cabri 3D installed. The teacher (second author) describes the students as motivated and comfortable working with each other and using both the DGEs and their dragging modality. The study took place towards the end of the second semester of the academic year in the participants' regular classrooms. At the time, the participants have learned about key invariants with respect to transformation groups. They did not have experienced with exploring 3D Euclidean geometrical concepts in class, nor before but in their junior high school studies.

Each group was first introduced to a new definition of quadrilateral that differs from the traditional one—a quadrilateral is a polygon with four sides—in that it adds what are the edges and faces of a quadrilateral. Second, the groups were given the task of drawing the bimedians of a quadrilateral and investigating the properties that the constructed objects satisfy, through the aid of Cabri II Plus. Then, for the purpose of comparing properties for quadrilaterals and for tetrahedra, the students were asked to discuss the movement to space using the *Redefinition* tool and the *Glassball* modality of Cabri 3D, about which they learned during this classroom.

One week later, the groups were given two main tasks: introducing the medians for quadrilaterals and tetrahedra, and conjecturing about the properties that hold in both cases. For the tasks involving bimedians and medians, the groups were given a Cabri 3D diagram that was new to them, in which a quadrilateral was drawn on the (grey) base plane—a plane given by default by the DGE. Finally, the students were asked to discuss possible proofs to show the validity of their claims in plane and in space.

Each day, two researchers (the authors) were present in class. They gave the groups the instructions and videotaped the group works with the DGEs and the classroom discussion at the end of the day. All written productions and DGEs' diagrams were collected. The aim was to analyse which kinds of visual skills the students were able to construct during the activity, whose relevant aspects are detailed below.

Quadrilaterals and Tetrahedra

The analogy between quadrilaterals and tetrahedra was established by introducing the definitions of edges and faces for a usual quadrilateral as follows: the segments joining two vertices of the quadrilateral are its *edges* and the triangles with vertices three vertices of the quadrilateral are its *faces*. In a quadrilateral there are six edges, the four sides and the two diagonals, and four faces, exactly as many as a tetrahedron.

So, we are able to look at—and *see*—figure ABCD, which has four vertices (A, B, C, D), six edges (AB, AB, BC, CD, DA, AC, BD) and four faces (ABC, ABD, ACD,

BCD), indifferently as the tetrahedron ABCD, whether we think that it lives in space, or as the quadrilateral ABCD, whether we think of it as a plane figure. We call *F* the figure ABCD. Given *F*, two *opposite edges* do not have common vertices and a face and a vertex are *opposite* when the vertex does not belong to the face. Moreover, we can define the *bimedians* and *medians* of *F* as: the segments that join the midpoints of two opposite edges; and the segments that join one vertex with the centroid of the opposite face, respectively. These objects satisfy some interesting properties:

- A. The three bimedians all pass through one point (that is, the centroid of *F*) that is the middle point of each bimedian.
- B. The four medians all meet in its centroid that divides each median in the ratio 1:3, the longest segment being on the side of the vertex of *F*.

As said above, the students were instructed to using the *Redefinition* tool and the *Glassball* modality of Cabri 3D. Let us make a thought experiment: imagine what would happen to a quadrilateral if one vertex was moved off the plane where it lies. The figure that was before ontologically became a new figure: the flat figure *is* now a solid figure; the polygon *is* now a polyhedron. It *is* as a consequence of the *movement* of the vertex off the plane.

The *Redefinition* tool realises this movement, redefining one point as an ontologically different point. For example, given the quadrilateral ABCD on the base plane (Figure 1a), one can *redefine* vertex D to be a point in space, exactly the apex D of the tetrahedron with base the face ABC (Figure 1b). The quadrilateral *is* a tetrahedron *within the DGE*, not necessarily in what the students see on the screen. The dragging and the *Glassball* modalities also allow for checking whether things "stay in synch"—in Hofstadter's words. In particular, the latter makes possible a rotation of the figure in order to actualise many virtual points of view from which to look at it.

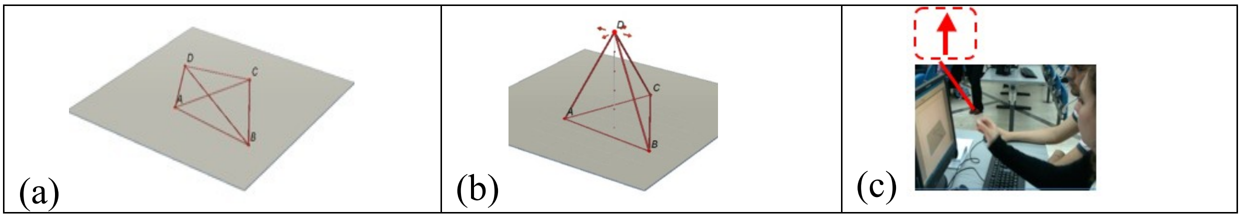

Figure 1(a) & (b): A quadrilateral ABCD on the base plane and the redefinition of point D for the tetrahedron ABCD. Figure 1(c): S gesturing the imagined movement.

DATA ANALYSIS

In this section, we present kinds of seeing in space that students developed in the tasks. The analysis is divided into two parts each containing a brief transcript and reference to the theoretical frame, for the purpose of identifying seeing in each part.

Seeing and Knowing

The following is a transcript of the initial interaction of three students, M, S and V, with the redefinition of one vertex of a quadrilateral using Cabri 3D. They had drawn without problems the four vertices and the six edges on the base plane. Before using the *Redefinition* tool, the teacher (T) proposed a thought experiment:

T: Let's try to extract a vertex from the plane. Imagine to redefine... rather, let's make two things. If I ask you: Extract a vertex from a plane, what do you think that it's gonna happen?

V: <*S gestures with two joined fingers moving up (Figure 1c)*> It shapes a kind of pyramid with a triangular base.

T: It becomes a pyramid with a triangular base. <*M drags a vertex to see the triangular pyramid, but as soon as he uses the Glassball modality he realises that the dragged point is on the plane. So, he drags the point back on the visible grey part of the base plane*>

V: If I extract the centre it becomes a square based pyramid.

T: What is the centre?

V: The point where the diagonals meet.

T: You find the Redefinition tool under the *Manipulation* button.

When the students were invited to "imagine" the new situation, gestures appeared that reflected knowledge about the transformation underwent by the quadrilateral. V easily imagined "a kind of pyramid with triangular base"; she was seeing the pyramid with her mind (lines 4-5). Instead, M immediately dragged the point to see the pyramid within the DGE. However, the *Glassball* modality, which he used to check the stability of the solid shape, revealed that "the dragged point is on the plane", contrary to M's expectation (lines 7-9). M wanted to see the guessed pyramid as a result of dragging, without knowing that a different tool was needed. Dragging only allows for moving the vertex on the base plane, even though it seems that it is off the plane once one sees it on the screen outside the visible grey part. This created a conflict between seeing and knowing that pushed M to drag the point back (line 8). At this time, V also imagined the solid obtained when "the centre" is extracted (line 10). However, since the students had no tools to check their conjectures, the teacher introduced the position of the *Redefinition* tool within the DGE.

Seeing something as something else

After the *Redefinition* tool was used to move the point and see the pyramid, the groups were given the 3D diagram with a quadrilateral on the base plane together with its bimedians. Slightly before knowing the task, M and S started to play around with the redefinition of one vertex. The teacher got close to ask them what happened:

M: It [the quadrilateral] becomes a tetrahedron. (…)

T: <*T talks about the bimedians*> Do they continue to meet?

M, S: Ya.

T: Where did they meet? <*T reads what M and S wrote before "the bimedians meet in a point H. H divides the bimedians into two equal parts"*> Does this still happen?

19 S: Hmm, at sight it seems to do, yea <*M and S looks at the figure on the screen*>

This brief extract shows that, thanks to the *Redefinition* tool, M and S came to see, and think of, the figure on the screen no longer as a quadrilateral but as a tetrahedron (line 14). This change was felt by the teacher, who drew attention to the continuity of the transformation (line 15), in order to push the student-pair to visually recognise ("at sight it seems", line 19) that what was happening before is actually an invariant under the transformation ("still happen", line 18). A similar reaction occurs when the two students explore the 3D diagram with a quadrilateral and its medians:

20 S: It's the same thing.
21 M: Ya, it is.
22 S: It's always upside-down. <*S refers to the tetrahedron with vertices the*
23 *centroids of the faces that they have constructed*>
24 M: "A" corresponds to "A'", and the others as well. [being A one vertex and A'
25 the centroid of the opposite face] The same properties hold.

Invariants were also grasped in the case of medians, in which seeing something as something else started to entail seeing the "the same thing" (line 20) in the figures and seeing "the same properties" holding for both (line 25). The use of "always" was significant here because it marked that the student-pair was generalizing (line 22). When the properties were discussed collectively, the students were given one final Cabri 3D file containing two figures that seemed to be exactly the same:

26 T: What are the figures? What do you see? <*Some students say "pyramids"*>
27 E: They're pyramids because we're using Cabri 3D.
28 C: No! The one on the left is a quadrilateral, the one on the right a pyramid.
29 T: How do you know?
30 C: I dragged the vertices, there are no projections on the left, yea on the right.
31 S: <*T asks "What about you?"*> The same, but we used the *Glassball*.
32 T: They might seem the same object, but they are not. We have seen that the
33 same definitions and properties hold for both figures. Then, does it
34 actually matter what they are? <*Students answer "No!"*>
35 Ss: No!

This last transcript clearly points out that, at the end, the students needed to use the resources of the DGE for distinguishing between the given figures: the fact of being within Cabri 3D does not really help to see them as different figures (lines 27-28). Instead, the dragging modality for seeing whether projections were present, or the *Glassball* tool for changing the point of view, gave them real answers (lines 30-31), even though it did no longer "matter what they are" (lines 35-37).

CONCLUDING REMARKS

Our examples showed that seeing in 3D involved for learners visual challenges that had mainly to do with conflicts between knowing and seeing and with the perception of the third dimension in flat diagrams. However, we found interesting the way these challenges were faced within the environment of Cabri 3D. The *Redefinition* tool and the *Glassball* modality, together with usual dragging, were resources for the students. In fact, they encouraged the students to take on multiple perspectives, as if they were taking on various physical positions from which to see a figure, as bodily projecting themselves both beyond and around it. So, they engaged the students in dynamic visual experiences with the diagrams, effecting new kinds of vision that pushed them towards a search for similarities and differences, invariants and changes, between quadrilaterals and tetrahedra. This engagement spoke directly to students' enhanced visual skills, so that they not only came to see the quadrilateral *as* a tetrahedron, but also to see them, when thought of as represented in a diagram, as "the same thing". For space constraints, we could not discuss our examples deeply, but we believe that they could form a basis for furthering effective future research.

References

Bakó, M. (2003). Different projecting methods in teaching spatial geometry. In M. A. Mariotti (Ed.), *Proc. 3rd Congress of the European Society for Research in Mathematics Education*. Bellaria, Italy: CERME.

Cutting, J. E. (1997). How the eye measures reality and virtual reality. *Behaviour Research Methods, Instruments, & Computers*, *29*(1), 27-36.

Healey, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Mathematical Thinking and Learning*, *1*(1), 59-84.

Hofstadter, D. R. (1997). Discovery and dissection of a geometric gem. In J. King & D. Schattschneider (Eds.), *Geometry turned on: Dynamic software in learning, teaching, and research* (pp. 3-14). Washington, DC: The Mathematical Association of America.

Hofstadter, D. R. (2005). On seeing A's and seeing As. In S. Franchi & G. Güzeldere (Eds.), *Mechanical bodies, computational minds* (pp. 249-264). Cambridge, MA: The MIT Press.

Parzysz, B. (1988). "Knowing" vs "seeing": Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics*, *19*(1), 79-92.

Poincaré, J. H. (1905). *Science and hypothesis*. London: Walter Scott.

Presmeg, N. (2006). Research on visualization in learning and teaching mathematics. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 205-235). Rotterdam/Tapei: Sense Publisher.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourse, and mathematizing*. Cambridge, MA: Cambridge University Press.

Whiteley, W. (2004, July). *Visualization in mathematics: Claims and questions towards a research program*. Paper presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark.